A harder 2 = 1 fallacy

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(A) Firstly, we like to prove:

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Method 1

Consider the infinite geometric series:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 + \cdots + \cdots \qquad \text{(first term is 1, common ratio is } -t\text{)}$$

where $|t| < 1$.

Integrate, we have $\int_0^x \frac{1}{1+t} dt = \int_0^x (1 - t + t^2 - t^3 + t^4 + \cdots + \cdots)\, dx$

$$[ln(1+t)]_0^x = \left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \cdots\right]_0^x$$

$$\therefore ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Method 2

Consider the Maclaurin series (Taylor series about the point 0) of the function:

$f(x) = ln(1+x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$ln(1+x)$	0	0
1	$\frac{1}{1+x}$	1	1
2	$-\frac{1}{(1+x)^2}$	-1	$-\frac{1}{2}$
3	$\frac{2}{(1+x)^3}$	2	$\frac{1}{3}$
4	$-\frac{3!}{(1+x)^3}$	$-3!$	$-\frac{1}{4}$

By Maclaurin series, $f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$

Using the table of derivatives above, we have

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

(B) Prove : $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$

Just put $x = 1$ in part **(A)** , we have our result.

(C) Prove : $2 = 1$.

Form **(B)**, we have $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \cdots$

Let us rearrange the right-hand side of the equality like this:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$\therefore \ln 2 = \frac{1}{2}\ln 2$$

$$\therefore 2 = 1$$

Where is the mistake of this fallacy?